LINEAR OPERATORS IN VH-SPACES

BY
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1. **Summary and introduction.** In [1] the concept of VH-spaces was introduced, and an almost exact analogue of a theorem of Sz.-Nagy [2] about linear operators in Hilbert space deduced as an application. (A linear space H is a VH-space if there is a vector "inner-product" defined for all pairs of elements of H, taking its values in a suitable topological vector space.)

In this paper we shall consider linear operators in VH-spaces more closely. The properties considered will be dictated by the main aim, which is to prove spectral representation theorems for certain classes of operators and to obtain a generalisation of Stone's theorem on one-parameter groups of unitary operators. Certain applications of these results will be given elsewhere [3].

We recall the definition of a VH-space. First, let Z be a complex (Hausdorff) topological vector space (TVS) and call Z *admissible* if it has the following properties:

(1) Z has an involution: i.e., a mapping $z \to z^*$ of Z onto itself with the properties
$$(z^*)^* = z, \quad (az_1 + bz_2)^* = \bar{a}z_1^* + \bar{b}z_2^*.$$

(2) There is a closed convex cone P in Z, such that $P \cap -P = 0$. We define a partial ordering in Z by writing $z_1 \geqq z_2$ if $z_1 - z_2 \in P$.

(3) The topology in Z is compatible with the partial ordering in the following sense: there is a basic set $\{N_\theta\}$ of convex neighbourhoods of the origin such that if $x \in N_\theta$ and $0 \leqq y \leqq x$ then $y \in N_\theta$. In particular Z is locally convex. Whenever a neighbourhood of the origin is mentioned it will tacitly be supposed to belong to this set $\{N_\theta\}$.

(4) If $x \geqq 0$, then $x^* = x$.

(5) Z is complete as a locally convex TVS.

If in addition Z has the following property it will be called *strongly admissible*.

(6) A bounded (with respect to the partial ordering) monotone sequence is convergent.

Then we have the following definition.

DEFINITION. A complex linear space E is a VE-space if there is a map $x, y \to [x, y]$ from $E \times E$ into an admissible space Z, with the following properties:

(i) $[x, x] \geqq 0$, and $[x, x] = 0$ if and only if $x = 0$.

(ii) $[x, y] = [y, x]^*$.

(iii) $[ax_1 + bx_2, y] = a[x_1, y] + b[x_2, y]$.

The map is called the (vector) inner product.

It was shown in [1] that if basic neighbourhoods U_θ of the origin in E are defined by

$$U_\theta = \{ x: [x, x] \in N_\theta \}$$

then E becomes a locally convex (Hausdorff) TVS, and the inner product is a continuous function on $E \times E$.

DEFINITION. A complex linear space H is a VH-space if it is a VE-space which is complete as a locally convex TVS. If the range space Z of the inner product is strongly admissible then we call H an LVH-space.

Function spaces which are closed under complex conjugation are VE-spaces with almost any topology, if the inner product is defined by pointwise multiplication

$$[f, g] = \overline{g} \cdot f.$$

Function spaces are usually VH-spaces, but not usually LVH-spaces.

Again, B^*-algebras are VH-spaces if the inner product is defined by $[x, y] = y^* x$.

We quote the following result from [1]: a VE-space E can be isomorphically imbedded as a dense subspace of a VH-space. In other words, a VE-space can be completed.

We have restricted attention to complex spaces. It is obviously possible to define real VH-spaces; we shall not, however, consider them here. The relationship between real and complex VH-spaces is much the same as that between real and complex Hilbert spaces, and our interest in spectral theorems for unitary operators is sufficient explanation of our decision to treat only the complex case.

Spaces with vector inner-products were also considered by Kaplansky [6], who imposed, however, considerably more algebraic structure; in particular the inner-product takes its values in an algebra. His aims were, of course, rather different but in any case the additional hypotheses made gave rise to a richer theory.

2. **Linear operators.** Clearly, we can define various classes of operators in a VH-space H, just as in a Hilbert space. For example, suppose A is a linear operator in H; if there exists an operator A^* such that for all $x, y \in H$

$$[Ax, y] = [x, A^*y]$$

we call A^* the adjoint of A. If it exists it is clearly unique and obeys the obvious rules. If $A = A^*$ we call A self-adjoint. If A^* exists and $AA^* = A^*A = I$, A is unitary, and if A^* exists and $AA^* = A^*A$, A is normal.

There is, of course, no reason why the adjoint of a general operator A should exist. Indeed it is easy to show that if a linear operator A in the function algebra $C[0,1]$ has an adjoint, A must correspond to multiplication by some fixed function $a(t) \in C[0,1]$.

We shall restrict our attention to continuous linear operators from now on, and in fact to a very special subclass of the continuous linear operators. We know that an operator A is continuous if given a neighbourhood N_θ of the origin in Z there exists another such neighbourhood N_ϕ with the property that

$$[x, x] \in N_\phi \Rightarrow [Ax, Ax] \in N_\theta.$$

Let us now define the class $\mathscr{B}$ of *bounded* operators by the condition that $A \in \mathscr{B}$ if there exists some constant k such that $[Ax, Ax] \leq k[x, x]$ for all $x \in H$. A bounded operator is clearly continuous.

For an operator $A \in \mathscr{B}$ we define the norm of A, $\|A\|$, as the positive square root of the minimum value of k for which

$$[Ax, Ax] \leq k[x, x].$$

It is easily seen that the set of values of k is closed, and we therefore have

$$[Ax, Ax] \leq \|A\|^2 [x, x].$$

THEOREM 1. *The class $\mathscr{B}$ forms a Banach algebra under the norm just defined.*

We must first show that $\|A\|$ is a norm in the usual sense; the only nontrivial step in the demonstration is the proof of the triangle inequality. However

$$[(A + B)x, (A + B)x] = [Ax, Ax] + [Bx, Bx] + [Ax, Bx] + [Bx, Ax]$$
$$\leq (\|A\|^2 + \|B\|^2)[x, x] + p[Ax, Ax] + p^{-1}[Bx, Bx]$$

for any $p > 0$, since for such p

$$[pAx - Bx, pAx - Bx]$$
$$= p^2[Ax, Ax] - p[Ax, Bx] - p[Bx, Ax] + [Bx, Bx] \geq 0.$$

Setting $p = \|B\| / \|A\|$, where we may clearly assume $\|A\| \neq 0$ since the result is trivially true if $\|A\| = 0$, we find

$$[(A + B)x, (A + B)x] \leq (\|A\| + \|B\|)^2[x, x]$$

and it follows that

$$\|A + B\| \leq \|A\| + \|B\|.$$

Clearly the multiplicative condition on the norm is satisfied, and $\|I\| = 1$, so that it remains only to show that $\mathscr{B}$ is complete.

Suppose that A_n is a Cauchy sequence in $\mathscr{B}$. Then

$$[(A_n - A_m)x, (A_n - A_m)x] \leqq \|A_n - A_m\|^2[x, x]$$

and the left-hand side tends to zero as n, m tend to infinity for each fixed x. Consequently A_nx is a Cauchy sequence in H, and converges to an element Ax. The operator A defined by this method is obviously linear, and the proof is completed as in a Banach space.

Now we can define various topologies for the linear operators in H, just as in a Hilbert space, provided the uniform topology is restricted to the class $\mathscr{B}$. Obviously the uniform topology is stronger than the strong topology, which is in turn stronger than the weak topology.

In [1] a contraction was defined as a linear operator T such that $[Tx, Tx] \leqq [x, x]$ for all $x \in H$. Contractions are thus those elements of $\mathscr{B}$ with norm less than or equal to unity. Furthermore any element of $\mathscr{B}$ is a scalar multiple of a contraction.

We denote the subset of $\mathscr{B}$ containing those bounded operators possessing adjoints by $\mathscr{B}^*$.

THEOREM 2. *If* $A \in \mathscr{B}^*$, $\|A\| = \|A^*\|$.

In particular, if an operator in $\mathscr{B}$ has an adjoint, the adjoint is itself in $\mathscr{B}$. This theorem in fact follows from Lemma 1 of [1], but we shall give the proof here.

We have

$$\begin{aligned}
2[A^*x, A^*x] &= 2[AA^*x, x] \\
&= [AA^*x, x] + [x, AA^*x] \\
&\leqq \|A\|^{-2}[AA^*x, AA^*x] + \|A\|^2[x, x] \\
&\leqq [A^*x, A^*x] + \|A\|^2[x, x].
\end{aligned}$$

Hence A^* is bounded, and $\|A^*\| \leqq \|A\|$, from which the result follows.

3. **Self-adjoint and positive operators in** $\mathscr{B}^*$. By definition an operator A is self-adjoint if and only if $[Ax, y] = [x, Ay]$ for all $x, y \in H$. We also have the following criterion.

LEMMA 1. *An operator* A *is self-adjoint if and only if*

$$[Ax, x] = [Ax, x]^*$$

for all $x \in H$.

The proof is exactly the same as that of the corresponding result in Hilbert space (see e.g. [4, p. 229]).

THEOREM 3. *If* $A \in \mathscr{B}^*$ *is self-adjoint, then*

$$- \|A\| [x,x] \le [Ax,x] \le \|A\| [x,x].$$

Conversely if for some operator A, and real numbers m, M,

$$m[x,x] \le [Ax,x] \le M[x,x],$$

where M, m are respectively the minimum and maximum values for which these inequalities are true, then $A \in \mathscr{B}^*$, A is self-adjoint, and $\|A\| = \max\{|m|,|M|\}$.

Proof.

$$2[Ax,x] = [Ax,x] + [x,Ax]$$

$$\le \|A\|^{-1}[Ax,Ax] + \|A\|[x,x]$$

$$\le 2\|A\|[x,x].$$

This gives one half of the first inequality, and the other is obtained by applying this result to $-A$.

Suppose then that

$$m[x,x] \le [Ax,x] \le M[x,x]$$

for some operator A. Then

$$[Ax,x] - m[x,x] \ge 0,$$

so that $[Ax,x] = [Ax,x]^*$ and by Lemma 1 A is self-adjoint.

Write $N_A = \max\{|m|,|M|\}$. Then following the argument on p. 230 of [4] we find

$$[Ax,Ax] \le \frac{1}{2} N_A\{\lambda^2[x,x] + \lambda^{-2}[Ax,Ax]\}$$

for any $\lambda > 0$. If $N_A = 0$, there is nothing left to prove; if $N_A > 0$, write $\lambda^2 = N_A$ and rearrange to find

$$[Ax,Ax] \le N_A^2[x,x].$$

Hence $A \in \mathscr{B}^*$ and $\|A\| \le N_A$, and the proof is completed by comparison with the first part of the theorem.

THEOREM 4. *$\mathscr{B}^*$ is a B^*-algebra.*

Clearly $\mathscr{B}^*$ is a normed algebra, since $\mathscr{B}^* \subset \mathscr{B}$. Suppose A is an arbitrary element of $\mathscr{B}^*$; then A^*A is self-adjoint. Hence

$$[Ax,Ax] = [A^*Ax,x] \le \|A^*A\| [x,x]$$

by Theorem 3, and thus $\|A\|^2 \le \|A^*A\|$, and it follows in the usual way that $\|A^*A\| = \|A\|^2$. It only remains to show that $\mathscr{B}^*$ is complete, and this is a direct consequence of the completeness of $\mathscr{B}$ and Theorem 2.

COROLLARY. *If* A ($\in \mathscr{B}^*$) *is a normal operator there exists a* (1-1) *correspondence between the algebra of all functions continuous on the spectrum of* A *and a subalgebra of* $\mathscr{B}^*$, *which is an isomorphism. In this correspondence the spectrum of* $f(A)$ *is the set of values assumed by* $f(\lambda)$.

As a B^*-algebra, $\mathscr{B}^*$ contains a cone of positive elements. There is, however, a natural definition of a positive operator A in H, as one which satisfies $[Ax, x] \geq 0$ for all $x \in H$, and it would be rather confusing if these definitions did not agree.

LEMMA 2. *Suppose* $A \in \mathscr{B}^*$. *Then* $[Ax, x] \geq 0$ *for all* $x \in H$ *if and only if* A *is positive as an element of the* B^*-algebra $\mathscr{B}^*$.

According to I, p. 312 of Naĭmark [5], an element A of a B^*-algebra with identity I is positive if it is self-adjoint and if

$$\| (\| A \| I - A) \| \leq \| A \|.$$

If $0 \leq [Ax, x] \leq \| A \| [x, x]$, then

$$0 \leq [(\| A \| I - A) x, x] \leq \| A \| [x, x],$$

and by Theorem 3 the criterion just mentioned is satisfied, so that A is positive as an element of the B^*-algebra.

Conversely, if A is positive in the B^*-algebra, $A = B^*B$ for some $B \in \mathscr{B}^*$ and

$$[Ax, x] = [Bx, Bx] \geq 0.$$

COROLLARY 1. *The spectrum of a self-adjoint operator* $A \in \mathscr{B}^*$ *lies in the interval* $[m, M]$ *(and includes the end points of this interval), where* m *and* M *are the quantities defined in Theorem 3.*

COROLLARY 2. *A positive operator* $A \in \mathscr{B}^*$ *has a unique positive square root* $A^{1/2} \in \mathscr{B}^*$.

COROLLARY 3. *The product of two positive operators in* $\mathscr{B}^*$ *which commute is itself positive.*

Next we show that $\mathscr{B}^*$ is closed under the formation of monotone limits, provided we deal with LVH-spaces.

THEOREM 5. *If* A_n *is a sequence of positive bounded operators in a* LVH-*space, with* $A_n \geq A_{n+1}$, *then* A_n *converges strongly to a bounded positive operator.*

Without loss of generality we may suppose that $A_n \leq I$ for all n. Now, by hypothesis, the sequence $[A_n x, x]$ is for fixed x a monotone decreasing sequence in Z, bounded below by 0, and hence converges. A fortiori it is a Cauchy sequence.

For any positive operator $A \in \mathscr{B}^*$, with $A \leq I$, we have

$$0 \leqq [A(x - Ax), x - Ax]$$
$$= [Ax, x] - [Ax, Ax] - [A^2x, x] + [A^2x, Ax]$$
$$\leqq [Ax, x] - 2[Ax, Ax] + [Ax, Ax],$$

so that $[Ax, x] \geqq [Ax, Ax] \geqq 0$.

If $m \leqq n$, $I \geqq A_m - A_n \geqq 0$, and applying the previous result

$$[(A_m - A_n)x, x] \geqq [(A_m - A_n)x, (A_m - A_n)x] \geqq 0.$$

The left-hand side converges to 0 as $m, n \to \infty$, so that $A_n x$ is a Cauchy sequence, and consequently converges to an element Ax. Clearly the operator A defined in this way is linear and we have, since $[A_n x, x]$ converges to $[Ax, x]$ for each x, and $0 \leqq [A_n x, x] \leqq [x, x]$,

$$0 \leqq [Ax, x] \leqq [x, x];$$

the conclusion therefore follows from Theorem 3.

4. Accessible subspaces and projections. A subspace M of a *VH*-space H was termed *accessible* in [1] if every element $w \in H$ can be decomposed into a sum

$$w = x + y$$

with $x \in M$ and y orthogonal to M in the sense that $[m, y] = 0$ for all $m \in M$. Clearly such a decomposition is unique if it is possible, and we can write $x = Pw$, where P is a linear operator, called the projection onto M. We have the following basic result.

THEOREM 6. *A projection P is self-adjoint and idempotent. Conversely, a self-adjoint idempotent operator is a projection onto its range subspace.*

COROLLARY. *A projection P is a positive contraction, with $[Px, x] = [Px, Px]$. Hence $P \in \mathscr{B}^*$ and an accessible subspace is closed.*

The proof is trivial.

The following propositions, in which P, P_1, P_2 are projections and M, M_1, M_2 are the corresponding subspaces, are easily proved in the same way as for Hilbert space.

(a) $0 \leqq P \leqq I$; $I - P$ is also a projection.

(b) If P_1 and P_2 commute, $P_1 P_2$ is the projection onto $M_1 \cap M_2$.

(c) If $P_1 P_2 = 0$, then M_1 and M_2 are orthogonal, and $P_1 + P_2$ is the projection onto $M_1 + M_2$.

(d) If P_1 and P_2 commute, $P_1 + P_2 - P_1 P_2$ is the projection onto $M_1 + M_2$.

(e) $P_1 \geqq P_2$ if and only if $P_1 P_2 = P_2$. In this case $M_2 \subset M_1$, and $P_1 - P_2$ is the projection onto the subspace of elements of M_1 which are orthogonal to M_2.

(f) In a LVH-space a monotonic sequence of projections is strongly convergent (to a projection).

Proposition (b) shows that two commuting projections have a greatest lower bound. The following proposition extends this to larger sets of commuting projections.

(g) A finite set of commuting projections in a VH-space has a greatest lower bound. A countable set of commuting projections in a LVH-space has a greatest lower bound.

Clearly these statements remain valid if "greatest lower bound" is replaced everywhere by "least upper bound."

5. The spectral representation theorem for bounded self-adjoint, unitary, and normal operators.

It is by now fairly obvious that the structure of VH-spaces is rich enough to support spectral decomposition theorems for suitable operators. We first define spectral measures and resolutions of the identity, exactly as in Hilbert space.

A spectral measure in a VH-space H is a set-function, defined for all sets of the field F generated by the rectangles in the complex plane, whose values are projections in H, which is (strongly) countably additive and has the identity operator as its value for the whole plane. If it has I as its value for some (smaller) set $S \in F$, it will be called a spectral measure on S. It is not clear that such a measure can be extended to all Borel sets in the plane.

A spectral measure on the real axis will be described by the point function E_λ, giving the value corresponding to the closed half-line $(-\infty, \lambda]$, and E_λ is then a resolution of the identity. That is, a resolution of the identity is a projection valued function E_λ defined on the real line such that:

(i) $E_\lambda \geq E_\mu$ if $\lambda \geq \mu$.

(ii) $E_\lambda \to E_\mu$ strongly as $\lambda \downarrow \mu$.

(iii) $E_\lambda \to I$ strongly as $\lambda \to +\infty$, and $E_\lambda \to 0$ strongly as $\lambda \to -\infty$.

(Such a function was called a *continuous* resolution of the identity in [1].)

Then we have the following results.

THEOREM 7. *If A is a bounded self-adjoint operator in a LVH-space, with $m[x, x] \leq [Ax, x] \leq M[x, x]$, then A has a (unique) spectral decomposition*

$$A = \int_{m-}^{M} \lambda dE_\lambda$$

where the integral is a Riemann-Stieltjes integral in the norm topology and E_λ is a resolution of the identity on the interval $[m, M]$.

THEOREM 8. *If U is a unitary operator in a LVH-space, U has a (unique) spectral decomposition*

$$U = \int_{0-}^{2\pi} e^{i\theta} dE_\theta$$

where the integral is defined in the norm topology and E_λ is a resolution of the identity on the interval $[0, 2\pi]$.

Theorem 9. *If N is a bounded normal operator in a LVH-space, N has a (unique) spectral decomposition*

$$N = \int \lambda E(d\lambda)$$

where the integral is defined in the norm topology and $E(\cdot)$ is a spectral measure on any set $\in F$ containing the spectrum of N (and a fortiori on any containing the disc $|\lambda| \leq \|A\|$).

The converses of these theorems, that integrals of these forms are self-adjoint, unitary, or normal bounded operators, are also true, and this largely accounts for our preoccupation with operators in $\mathscr{B}^*$.

The proofs of these theorems for operators in Hilbert space given in [4, pp. 272-275, 280-284, 286-288] can be transferred word for word to the present situation.

6. A generalisation of Stone's theorem. It follows at once from Theorem 8 that if U is a unitary operator in a LVH-space

$$U^n = \int_0^{2\pi} e^{in\theta} dE_\theta \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

In this section Stone's theorem on the spectral representation of a group of unitary operators with a continuous parameter t $(-\infty < t < \infty)$ will also be shown to be true in the present context. Presumably the general theorem of this type, on the unitary representations of an arbitrary Abelian locally compact group, can also be extended if the conditions on Z are suitably strengthened, although it appears rather doubtful whether the standard proofs of this result can be adapted (cf. [5, p. 419] and the references on p. 392 of [4]).

Theorem 10. *Let U_t $(-\infty < t < \infty)$ be a weakly continuous group of unitary operators in a LVH-space H. Then there exists a (unique) resolution of the identity E_λ such that for each t and each $x \in H$*

$$U_t x = \int_{-\infty}^\infty e^{i\lambda t} dE_\lambda x,$$

where the integral is a Riemann-Stieltjes integral. It is obvious that conversely for any resolution of the identity these integrals define a weakly continuous group of unitary operators.

Proof. The proof of Stone's theorem given in [4, pp. 381-383] can be transferred almost without change. Consequently we shall only remark on

those points which are not immediately obvious.

We begin by showing that if $U_1 = I$ the linear operators P_n, defined as a Riemann integral for each $x \in H$ by

$$P_n x = \int_a^{a+1} e^{-2\pi i n t} U_t x \, dt$$

are a set of mutually orthogonal projections, with sum I.

The only point requiring any attention is the proof that if

$$\int_0^1 e^{-2\pi i n t} U_t Q x \, dt = 0 \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

then $U_t Q x = 0$, where $Q = I - \sum_{-\infty}^{\infty} P_n$ is a projection.

However, if f is an arbitrary continuous linear functional in H, it follows that

$$\int_0^1 e^{-2\pi i n t} f(U_t Q x) \, dt = 0$$

and consequently $f(U_t Q x)$, being a continuous scalar-valued function, vanishes for all t. Since H is a locally convex TVS there are enough linear functionals to separate points, hence $U_t Q = 0$, and the argument of [4] can be resumed.

There are no other points in the proof which require special attention.

7. **Ergodic theorems.** As in Hilbert space ergodic theorems can be deduced from Theorems 8 and 10.

THEOREM 11. *Let U be a unitary transformation of the LVH-space H. Then for each $x \in H$*

$$\frac{1}{n - m} \sum_{k=m}^{n-1} U^k x$$

converges as $n - m \to \infty$ to an element x^* invariant under U.

THEOREM 12. *Let U_t ($-\infty < t < \infty$) be a weakly continuous group of unitary operators in the LVH-space H. Then for each $x \in H$*

$$\frac{1}{T} \int_a^{a+T} U_t x \, dt$$

converges as $T \to \infty$ to an element x^* invariant under the operators U_t.

The limit element x^* can be identified in both cases as $\Delta E_0 x$, where $\Delta E_0 = E_0 - E_{0-}$ is the jump at the origin in the resolution of the identity associated with U or U_t.

COROLLARY. *Let T_t ($t \geq 0$, with $T_0 = I$) be a weakly continuous semigroup of contractions with adjoints in a LVH-space H. Then for each $x \in H$ and $a \geq 0$*

$$\frac{1}{T} \int_a^{a+T} T_t x \, dt$$

converges as $T \to \infty$ to an element invariant under each T_t. Similarly if T is a contraction with an adjoint in a LVH-space H

$$\frac{1}{n-m} \sum_{k=m}^{n-1} T^k x$$

converges as $n - m \to \infty$ to an element invariant under T.

The corollary follows from the theorems via an application of Theorems 6 and 7 of [1]. (Cf. the corresponding proof for Hilbert space in [2].)

We shall give the proof of Theorem 12. The proof of Theorem 11 is obtained by replacing the appropriate integral by a sum.

According to Theorem 10, we have

$$\frac{1}{T} \int_a^{a+T} U_t x \, dt = \frac{1}{T} \int_a^{a+T} dt \int_{-\infty}^{\infty} e^{i\lambda t} dE_\lambda x.$$

Both integrals are of Riemann or Riemann-Stieltjes type, and it is not difficult to show that the order of integration can be reversed. Then the right-hand side becomes

$$\int_{-\infty}^{\infty} e^{i\lambda a} \frac{(e^{i\lambda T} - 1)}{i\lambda T} \, dE_\lambda x, = \Delta E_0 x + \int_{-\infty}^{\infty} e^{i\lambda a} \frac{(e^{i\lambda T} - 1)}{i\lambda T} \, dE_\lambda x,$$

where the bar on the integral sign signifies the omission of the value 0. Denote the integrand by $\phi(a, \lambda, T)$. Then $|\phi(a, \lambda, T)| \leq 1$ while uniformly in $|\lambda| > \delta$

$$|\phi(a, \lambda, T)| \leq 2/\delta T.$$

Let N_θ be an arbitrary neighbourhood of the origin in Z, and N_ϕ a neighbourhood such that $N_\phi + N_\phi \subset N_\theta$.

We have

$$\left[\int_{-\infty}^{\infty} \phi(a, \lambda, T) dE_\lambda x, \int_{-\infty}^{\infty} \phi(a, \lambda, T) dE_\lambda x \right]$$

$$= \int_{-\infty}^{\infty} |\phi(a, \lambda, T)|^2 d[E_\lambda x, x]$$

$$= \int_{-\delta}^{\delta} |\phi(a, \lambda, T)|^2 d[E_\lambda x, x] + \int_{|\lambda| > \delta} |\phi(a, \lambda, T)|^2 d[E_\lambda x, x].$$

Since the integrand is bounded by unity, the first integral is no greater than

$$[E_\delta x, x] - [E_0 x, x] + [E_{0-} x, x] - [E_{-\delta-} x, x]$$

which is included in N_ϕ if δ is chosen sufficiently small.

The second integral is no greater than

$$4/\delta^2 T^2 [x, x]$$

which is included in N_ϕ for all sufficiently large T. The theorem follows.

8. **Fourier-Stieltjes representations of positive-definite functions.** It is not altogether unexpected that Theorems 8 and 10 give rise to representations of positive-definite functions, defined in a suitable sense, analogous to those known for complex-valued positive-definite functions.

Suppose ζ_n $(n = 0, \pm 1, \pm 2, \cdots)$ is a sequence of elements in an admissible space Z. Then it is positive-definite if

$$\sum_{i,j} c_i \overline{c_j} \zeta_{i-j} \geqq 0$$

for every finite set of complex numbers c_i.

Similarly, if ζ_t is a function of the real parameter t $(-\infty < t < \infty)$ with values in Z, it is positive-definite if

$$\sum_{i,j} c_i \overline{c_j} \zeta_{t_i - t_j} \geqq 0$$

for every finite set of values t_i and complex numbers c_i.

THEOREM 13. *If ζ_n is a positive-definite sequence in a strongly admissible space Z, there is a monotone increasing positive function $m(\lambda)$ of the real parameter λ, which takes its values in Z and is continuous on the right, such that*

$$\zeta_n = \int_{0-}^{2\pi} e^{in\lambda} dm(\lambda) \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Similarly, if ζ_t is a continuous positive-definite function in a strongly admissible space Z, there is a monotone increasing positive function $m(\lambda)$ of the real parameter λ, which takes its values in Z, is continuous on the right, and has $m(\lambda) \to 0$ as $\lambda \to -\infty$ and $m(\lambda) \to \zeta_0$ as $\lambda \to +\infty$, such that

$$\zeta_t = \int_{-\infty}^{\infty} e^{i\lambda t} dm(\lambda) \qquad (-\infty < t < \infty).$$

We shall give the proof for positive-definite sequences. The continuous parameter version is proved in the same way.

Let E be the space of all sequences of elements in Z of the form

$$z_n = \sum_m c_m \zeta_{m-n}$$

where the (complex) coefficients c_m are nonzero for only a finite number of indices. If $x = \{\sum_m c_m \zeta_{m-n}\}$ and $y = \{\sum_m d_m \zeta_{m-n}\}$ define the inner-product in E by

$$[x, y] = \sum_{m,n} c_m \bar{d}_n \zeta_{m-n}.$$

It is easy to show that this is a valid definition of an inner-product, and in fact from now on the construction is almost identical with that in [1] and [2]. The space E can be completed to an LVH-space H, and the operators D_k, defined in E by

$$D_k \left\{ \sum_m c_m \zeta_{m-n} \right\} = \left\{ \sum_m c_{m-k} \zeta_{m-n} \right\}$$

form a unitary representation in E of the additive group of integers, and being in particular continuous can be extended to H as a representation of the same group. It follows that

$$D_k = \int_{0-}^{2\pi} e^{ik\lambda} dE_\lambda$$

by Theorem 8, and hence

$$\zeta_k = [D_k u, u] = \int_{0-}^{2\pi} e^{ik\lambda} dm(\lambda)$$

where u is the element $\{\zeta_{-n}\}$ of E and $m(\lambda) = [E_\lambda u, u]$.

For sequences of operators in an LVH-space which are positive-definite in the sense of [1], a similar representation can be obtained. A corresponding theorem for functions of a continuous parameter is also valid.

THEOREM 14. *Let T_n $(n = 0, \pm 1, \pm 2, \cdots)$ be a sequence of continuous linear operators in an LVH-space H, for which $T_0 = I$ and*

$$\sum_{m,n} [T_{m-n} x_m, x_n] \geqq 0$$

for an arbitrary finite set of elements $x_n \in H$. Then

$$T_n = \int_{0-}^{2\pi} e^{in\theta} dB(\theta)$$

where $B(\theta)$ is a monotone increasing positive operator-valued function of θ, continuous on the right, equal to 0 at $0-$ and to I at 2π.

COROLLARY. *Let T_n be a sequence of continuous linear operators in a Hilbert space H. Then $\sum c_m \bar{c}_n T_{m-n} \geq 0$ for all finite sets of complex constants c_n if and only if*

$$\sum (T_{m-n} x_m, x_n) \geqq 0$$

for all finite sets of elements $x_n \in H$.

It will be observed that $B(\theta)$ is essentially a generalised (continuous) resolution of the identity, as defined in [1] (following the usual definition in Hilbert space, cf. [2]).

Proof. It was shown in [1] that under these hypotheses T_n can be represented as the restriction to H of the operator PD_n in some larger LVH-space $\hat{H}$, where P is the projection onto H and $\{D_n\}$ is a unitary representation of the additive group of integers. It follows from Theorem 8 that T_n is the restriction to H of

$$\int_{0-}^{2\pi} e^{in\theta} d(PE_\theta),$$

and the restriction $B(\theta)$ of PE_θ to H is of just the type described in the theorem.

The corollary follows almost immediately from a comparison of Theorems 13 and 14, since the continuous operators in a Hilbert space form a strongly admissible space with the weak topology. The only nontrivial point in the proof of this is in showing that the topology and the partial ordering are compatible. However, it is easy to show that sets such as $\{T: |(Tx, x)| \leqq \epsilon\}$ form a subbase for the weak operator topology, and the compatibility becomes apparent.

9. **Acknowledgement.** I am indebted to a referee for the reference to Kaplansky's work.

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